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A Simple Proof of an Estimate for the Approximation of the Euclidean Ball and the Delone Triangulation Numbers

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We give a simple proof of an estimate for the approximation of the Euclidean ball by a polytope with a given number of vertices with respect to the volume of the symmetric difference metric and relatively precise estimate for the Delone triangulation numbers. We also study the same problem for a given number of $n-1$-dimensional faces. © 2000 Academic Press

In this note we present a simple proof of an estimate for the approximation of a convex body by a polytope due to Gordon, Reisner, and Schütt [GRS].

By B_2^n we denote the Euclidean ball in $\mathbb{R}^n$. Recall that the Hausdorff distance between two convex bodies K and C is defined by

$$\mathrm{d}_H(K, C) = \max\{\max_{x \in C} \min_{y \in K} \|x-y\|, \max_{y \in K} \min_{x \in C} \|x-y\|\},$$

where $\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^n$, and that the symmetric difference metric is the volume of the symmeric difference of K and C,

$$\mathrm{d}_S(K, C) = \mathrm{vol}_n(K \triangle C).$$

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Bronshteyn and Ivanov, [BI], proved that there are absolute constants c_1 and c_2 such that for every convex body K contained in the Euclidean unit ball B_2^n and for every sufficiently small $\varepsilon > 0$ there is a polytope P_ε contained in K with the number of vertices not greater than $c_1 \sqrt{n}\,(c_2/\varepsilon)^{(n-1)/2}$ such that

$$d_H(P_\varepsilon, K) \leqslant \varepsilon.$$

This implies the existence of a constant c_3 such that for every $n \in \mathbb{N}$ and every convex body K in $\mathbb{R}^n$, and every $N \in \mathbb{N}$ there is a polytope P_N contained in K with N vertices such that

$$\mathrm{vol}_n(K) - \mathrm{vol}_n(P_N) \leqslant c_3\, n\, \mathrm{vol}_n(K)\, N^{-2/(n-1)}. \tag{1}$$

On the other hand, Macbeath [Mac] showed that the Euclidean ball is the most difficult convex body to approximate by a polytope in the symmetric difference metric. More precisely, he proved that for every convex body K in $\mathbb{R}^n$ with $\mathrm{vol}_n(K) = \mathrm{vol}_n(B_2^n)$ we have

$$\begin{aligned} &\inf\{\mathrm{d}_S(K, P_N) \mid P_N \subset K \text{ and } P_N \text{ has at most N vertices}\} \\ &\qquad \leqslant \inf\{\mathrm{d}_S(B_2^n, P_N) \mid P_N \subset B_2^n \text{ and } P_N \text{ has at most N vertices}\}. \end{aligned}$$

Thus, in order to decide whether the estimate (1) is optimal, it suffices to study the case of the Euclidean ball. This has been done by Gordon, Reisner and Schütt in [GRS]. Namely, they proved that there is a constant c_4 such that for every polytope $P_N \subset B_2^n$ with at most N vertices the following inequality holds

$$\mathrm{vol}_n(B_2^n) - \mathrm{vol}_n(P_N) \geqslant c_4\, n\, \mathrm{vol}_n(B_2^n)\, N^{-2/(n-1)}. \tag{2}$$

Shortly after the paper [GRS] had been written, the third named author presented it at the Academy of Sciences in Warsaw. Then the first named author of this paper suggested a simpler way to prove the estimate.

Gruber, [Gr], obtained an asymptotic formula for convex bodies K in $\mathbb{R}^n$ with a C^2-boundary with everywhere positive curvature. Namely, for such bodies

$$\inf\{\mathrm{d}_S(K, P_N) \mid P_N \subset K \text{ and } P_N \text{ has at most N vertices}\}$$

is asymptotically equivalent to

$$\frac{1}{2}\,\mathrm{del}_{n-1}\left(\int_{\partial K} \kappa(x)^{1/(n+1)}\, d\mu(x)\right)^{(n+1)/(n-1)} \frac{1}{N^{2/(n-1)}},$$

where μ is the surface measure, κ the Gauß-curvature, and del_{n-1} is a constant connected with Delone triangulations. We comment briefly on the Delone triangulation [Ed, Gr]. Let D be a finite subset of $\mathbb{R}^{n-1}$ that is not contained in an affine subspace of lower dimension. The Delone triangulation of D is the unique tiling of the convex hull of D with proper convex polytopes, each having the following property: Its vertices belong to D and are on the boundary of a $n-1$-dimensional Euclidean ball which contains no further point of D. Let Pb be the paraboloid

$$\left\{x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n-1} x_i^2 \leqslant x_n\right\}$$

and let Q be a proper convex polytope inscribed in Pb. It is not difficult to show that the orthogonal projections of the facets of Q that are on the lower side of Q are a Delone triangulation of the orthogonal projection of the vertices of Q to $\mathbb{R}^{n-1}$. This construction is used to define del_{n-1}. Thus

$$\lim_{N\to\infty} \frac{2\inf\{\mathrm{d}_S(K, P_N) \mid P_N \subset K \text{ and } P_N \text{ has at most N vertices}\}}{\left(\int_{\partial K} \kappa(x)^{1/(n+1)}\, d\mu(x)\right)^{(n+1)/(n-1)} N^{-2/(n-1)}} = \mathrm{del}_{n-1}.$$

In particular, for $K = B_2^n$, we get

$$\lim_{N\to\infty} \frac{2\inf\{\mathrm{d}_S(B_2^n, P_N) \mid P_N \subset B_2^n \text{ and } P_N \text{ has at most N vertices}\}}{(\mathrm{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)} N^{-2/(n-1)}} = \mathrm{del}_{n-1}. \tag{3}$$

Hence, by (1) and (2), there are constants c_5 and c_6 [GRS] such that

$$c_5 n \leqslant \mathrm{del}_n \leqslant c_6 n\,.$$

In this paper we present quite precise estimates for the constants c_5 and c_6.

Let $N(n, \phi)$ denote the maximal number of vectors x_i, $i = 1, ..., M$ in ∂B_2^n satisfying

$$\langle x_i, x_j\rangle \leqslant \cos\phi,$$

for $i \neq j$.

Kabatjanskii and Levenstein [KL] showed that

$$N(n, \phi) \leqslant (1-\cos\phi)^{-n/2}\, 2^{-0.901n}. \tag{4}$$

For a fixed finite subset $x_1, x_2, ..., x_N$ of ∂B_2^n such that the polytope $P_{[x_1, ..., x_N]}$ spanned by it contains the origin as an interior point we define the function t: $[0, \pi] \to [0, 1]$ by

$$\mathrm{vol}_{n-1}\{x \in \partial B_2^n \mid \max_{1 \leqslant i \leqslant N} \langle x, x_i \rangle \geqslant \cos \theta\} = t(\theta) \,\mathrm{vol}_{n-1}(\partial B_2^n). \tag{5}$$

Obviously the function t is increasing. Moreover, it is continuous. Indeed, for $\theta < \eta$ we have

$$\begin{aligned}
0 &\leqslant (t(\eta) - t(\theta)) \,\mathrm{vol}_{n-1}(\partial B_2^n) \\
&= \mathrm{vol}_{n-1}\left(\bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\eta}{2}\right)\right)\right) \\
&\quad - \mathrm{vol}_{n-1}\left(\bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\theta}{2}\right)\right)\right) \\
&= \mathrm{vol}_{n-1}\left(\bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\eta}{2}\right)\right) \Big\backslash \bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\theta}{2}\right)\right)\right) \\
&\leqslant \sum_{i=1}^{N} \mathrm{vol}_{n-1}\left(\partial B_2^n \cap \left(B_2^n\left(x_i, 2 \sin \frac{\eta}{2}\right) \Big\backslash B_2^n\left(x_i, 2 \sin \frac{\theta}{2}\right)\right)\right) \\
&= N \,\mathrm{vol}_{n-1}\left(\partial B_2^n \cap \left(B_2^n(x_1, 2 \sin \frac{\eta}{2}\right) \Big\backslash B_2^n\left(x_1, 2 \sin \frac{\theta}{2}\right)\right)\right).
\end{aligned}$$

Clearly the last expression can be made as small as required provided that θ is sufficiently close to η.

Thus there is a smallest number θ_0 such that $t(\theta_0) = 1$. Now we restrict the function t to the interval $[0, \theta_0]$. We claim that the function t is on this interval strictly increasing. To verify this let $\theta < \eta \leqslant \theta_0$. Since θ_0 is the smallest number with $t(\theta_0) = 1$ we infer that $t(\theta) < 1$. Hence

$$\left(\bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\theta}{2}\right)\right)\right)^c$$

is an open, nonempty set. Moreover, there are

$$y \in \left(\bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\theta}{2}\right)\right)\right)^c$$

and $\varepsilon > 0$ such that

$$\partial B_2^n \cap B_2^n(y, \varepsilon) \subseteq \bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n\left(x_i, 2 \sin \frac{\eta}{2}\right)\right)$$

and

$$\partial B_2^n \cap B_2^n(y, \varepsilon) \subseteq \left(\bigcup_{i=1}^{N} \left(\partial B_2^n \cap B_2^n \left(x_i, 2 \sin \frac{\theta}{2} \right) \right) \right)^c.$$

Thus $t(\theta) < t(\eta)$.

Altogether, we get that $t: [0, \theta_0] \to [0, 1]$ is a continuous, strictly increasing function onto the unit interval. Therefore, its inverse function $\theta: [0, 1] \to [0, \theta_0]$ is an increasing continuous function with $\theta(0) = 0$.

LEMMA 1. *Let $x_i \in \partial B_2^n$ for $i = 1, ..., N$ and let P_N denote the convex hull of the points $x_1, x_2, ..., x_N$. Assume that that $0 \in \operatorname{int} P_N$. Then*

$$\operatorname{vol}_n(B_2^n) - \operatorname{vol}_n(P_N) \geqslant \operatorname{vol}_n(B_2^n) \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} \times \frac{1}{2^k} \frac{n-1}{2k+n-1} \left(\cos \theta(1) \frac{n}{N} \frac{\operatorname{vol}_n(B_2^n)}{\operatorname{vol}_{n-1}(B_2^{n-1})} \right)^{2k/(n-1)},$$

where $\theta(t)$ is the inverse function of the function $t(\theta)$ given by (5).

Proof. First note that $0 \in \operatorname{int} P_N$ implies that $\theta(1) < \frac{\pi}{2}$, where θ is the function defined above. Thus, we have

$$\begin{aligned} t \operatorname{vol}_{n-1}(\partial B_2^n) &= \operatorname{vol}_{n-1} \left(\bigcup_{i=1}^{N} (\partial B_2^n \cap B_2^n(x_i, \sqrt{2 - 2 \cos \theta(t)})) \right) \\ &\leqslant N \operatorname{vol}_{n-2}(\partial B_2^{n-1}) \int_0^{\theta(t)} \sin^{n-2}\phi \, d\phi \\ &= N \operatorname{vol}_{n-2}(\partial B_2^{n-1}) \int_0^{\sin \theta(t)} \frac{s^{n-2}}{\sqrt{1-s^2}} \, ds \\ &\leqslant N \operatorname{vol}_{n-2}(\partial B_2^{n-1}) \frac{1}{\cos \theta(t)} \int_0^{\sin \theta(t)} s^{n-2} \, ds \\ &= N \operatorname{vol}_{n-2}(\partial B_2^{n-1}) \frac{1}{n-1} \frac{\sin^{n-1} \theta(t)}{\cos \theta(t)}. \end{aligned}$$

This implies

$$\sin^{n-1}\theta(t) \geqslant t \frac{n}{N} \cos \theta(1) \frac{\operatorname{vol}_n(B_2^n)}{\operatorname{vol}_{n-1}(B_2^{n-1})}.$$

Let σ be the normalized measure on ∂B_2^n. For $x \in \partial B_2^n$ let $r(x)$ be the distance of 0 to the point of intersection of $[0, x]$ and P_N. Then

$$\mathrm{vol}_n(P_N) = \mathrm{vol}_n(B_2^n) \int_{\partial B_2^n} r(x)^n \, d\sigma(x).$$

Since $r(x) \leqslant \max_{1 \leqslant i \leqslant N} \langle x, x_i \rangle$ we get

$$\mathrm{vol}_n(P_N) \leqslant \mathrm{vol}_n(B_2^n) \int_{\partial B_2^n} \left(\max_{1 \leqslant i \leqslant N} \langle x, x_i \rangle \right)^n d\sigma(x).$$

For a partition $\{t_0, t_1, ..., t_m\}$ of $[0, 1]$ we put

$$A_j = \{ x \in \partial B_2^n \mid \cos \theta(t_{j+1}) < \max_{1 \leqslant i \leqslant N} \langle x, x_i \rangle \leqslant \cos \theta(t_j) \}.$$

For every $\varepsilon > 0$ there is a partition so that

$$\left| \int_{\partial B_2^n} \left(\max_{1 \leqslant i \leqslant N} \langle x, x_i \rangle \right)^n d\sigma(x) - \int_{\partial B_2^n} \sum_{j=1}^m \chi_{A_j} \cos^n \theta(t_{j+1}) \, d\sigma(x) \right| < \varepsilon.$$

On the other hand

$$\int_{\partial B_2^n} \sum_{j=1}^m \chi_{A_j} \cos^n \theta(t_{j+1}) \, d\sigma(x) = \sum_{j=1}^m (t_{j+1} - t_j) \cos^n \theta(t_{j+1}).$$

The last expression is a Riemann sum for the integral

$$\int_0^1 \cos^n (\theta(t)) \, dt.$$

Thus we get

$$\begin{aligned}
\mathrm{vol}_n(P_N) &\leqslant \mathrm{vol}_n(B_2^n) \int_0^1 \cos^n (\theta(t)) \, dt \\
&\leqslant \mathrm{vol}_n(B_2^n) \int_0^1 \left(1 - \frac{1}{2} \sin^2 \theta(t) \right)^n dt \\
&\leqslant \mathrm{vol}_n(B_2^n) \int_0^1 \left(1 - \frac{1}{2} \left(t \frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)} \right)^n dt \\
&= \mathrm{vol}_n(B_2^n) \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{2^k} \qquad (6) \\
&\quad \times \left(\frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2k/(n-1)} \int_0^1 t^{2k/(n-1)} \, dt. \quad \blacksquare
\end{aligned}$$

LEMMA 2. *Let $C(\rho, \Delta)$ be a cap of height Δ of the Euclidean ball of radius ρ. Then*

$$2(2\rho)^{(n-1)/2} \frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n+1} \left\{ \Delta^{(n+1)/2} - \frac{n^2-1}{4\rho(n+3)} \Delta^{(n+3)/2} \right\} \leqslant \mathrm{vol}_n(C(\rho, \Delta)).$$

Proof.

$$\mathrm{vol}_n(C(\rho, \Delta)) = \mathrm{vol}_{n-1}(B_2^{n-1}) \int_0^{\Delta} (2\rho t - t^2)^{(n-1)/2} \, dt$$
$$= (2\rho)^{(n-1)/2} \mathrm{vol}_{n-1}(B_2^{n-1}) \int_0^{\Delta} t^{(n-1)/2} \left(1 - \frac{t}{2\rho}\right)^{(n-1)/2} dt$$

Since $(1-u)^{\alpha} \geqslant 1 - \alpha u$ for $0 \leqslant u \leqslant 1$ and $\alpha \geqslant 1$ we get

$$\mathrm{vol}_n(C(\rho, \Delta)) \geqslant (2\rho)^{(n-1)/2} \mathrm{vol}_{n-1}(B_2^{n-1}) \int_0^{\Delta} t^{(n-1)/2} - \frac{n-1}{4\rho} t^{(n+1)/2} \, dt$$
$$= (2\rho)^{(n-1)/2} \mathrm{vol}_{n-1}(B_2^{n-1})$$
$$\times \left\{ \frac{2}{n+1} \Delta^{(n+1)/2} - \frac{n-1}{2\rho(n+3)} \Delta^{(n+3)/2} \right\}. \quad \blacksquare$$

THEOREM 3. *The following inequality holds*

$$\frac{n-1}{n+1} \mathrm{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)} \leqslant \mathrm{del}_{n-1} \leqslant 2^{0.802} \mathrm{vol}_{n-1}(\partial B_2^n)^{-2/(n-1)},$$

for every $n \in \mathbb{N}$.

Proof. By compactness, for each N there are points $x_1, ..., x_N \in \partial B_2^n$ such that

$$\mathrm{vol}_n(B_2^n) - \mathrm{vol}_n(P_N)$$
$$= \inf\{ \mathrm{d}_S(B_2^n, P_N) \mid P_N \subset B_2^n \text{ and } P_N \text{ has at most N vertices} \},$$

where $P_N = P_{[x_1, ..., x_N]}$. For each such subset let $\theta_N(t)$ be the inverse of the function defined by (5).

By (3), we have

$$\mathrm{del}_{n-1} = \lim_{N \to \infty} 2 \frac{\mathrm{d}_S(B_2^n, P_N)}{N^{-2/(n-1)} (\mathrm{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)}}.$$

It follows from Lemma 1 that

$$\lim_{N\to\infty} N^{2/(n-1)}(\mathrm{vol}_n(B_2^n)-\mathrm{vol}_n(P_N))$$
$$\geqslant \lim_{N\to\infty} \frac{n-1}{2(n+1)}(\cos\theta_N(1))^{2/(n-1)} \frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)}}{(\mathrm{vol}_{n-1}(B_2^{n-1}))^{2/(n-1)}}.$$

We claim that $\lim_{N\to\infty}\theta_N(1)=0$. Indeed, suppose that this is not the case. Then there is $N_0\in\mathbb{N}$ and $\alpha>0$ such that for every $N\geqslant N_0$ there exists $x\in\partial B_2^n$ whose angular distance to all x_i, for $i=1, ..., N$, is not less than α. This implies that there is a cap of spherical radius α whose interior has an empty intersection with P_N. Thus, by Lemma 2, we get

$$\mathrm{vol}_n(B_2^n)-\mathrm{vol}_n(P_N)\geqslant 2^{(n+1)/2}\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{n+1}\left\{\Delta^{(n+1)/2}-\frac{n^2-1}{4(n+3)}\Delta^{(n+3)/2}\right\},$$

where the height $\Delta=\sin\alpha\ \tan(\frac{\pi}{2}-\frac{\alpha}{2})$. Since the right hand side of the inequality does not depend on N we get a contradiction with (1). Therefore

$$\lim_{N\to\infty} N^{2/(n-1)}(\mathrm{vol}_n(B_2^n)-\mathrm{vol}_n(P_N))\geqslant\frac{n-1}{2(n+1)}\frac{(\mathrm{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)}}{(\mathrm{vol}_{n-1}(B_2^{n-1}))^{2/(n-1)}},$$

and consequently

$$\mathrm{del}_{n-1}\geqslant\frac{n-1}{n+1}\mathrm{vol}_{n-1}(B_2^{n-1})^{-2/(n-1)}.$$

To prove the right hand side inequality note that, by the result of Kabatjanskii and Levenstein, for a given angle ϕ there exist points $x_1, ..., x_N$ such that

$$\cos\phi\geqslant\langle x_i, x_j\rangle \qquad \text{for all}\quad i\neq j$$
$$\forall x\in\partial B_2^n\ \exists i:\cos\phi\leqslant\langle x, x_i\rangle$$
$$N\leqslant(1-\cos\phi)^{-(n-1)/2}\,2^{0.901(n-1)}.$$

Set $P_N=[x_1, ..., x_N]$. Then

$$P_N\supseteq\left(\cos\frac{\phi}{2}\right)B_2^n.$$

Since $\cos t \geqslant 1 - \frac{1}{2}t^2$, we get

$$\begin{aligned}\operatorname{vol}_n(B_2^n) - \operatorname{vol}_n(P_N) &\leqslant \operatorname{vol}_n(B_2^n)\left(1 - \left(\cos\frac{\phi}{2}\right)^n\right)\\ &\leqslant \operatorname{vol}_n(B_2^n)\left(1 - \left(1 - \frac{1}{2}\left(\frac{\phi}{2}\right)^2\right)^n\right)\\ &\leqslant \frac{n}{8}\phi^2 \operatorname{vol}_n(B_2^n).\end{aligned}$$

Since

$$2^{1.802} N^{-2/(n-1)} \geqslant 1 - \cos\phi \geqslant \tfrac{1}{2}\phi^2 - \tfrac{1}{24}\phi^4$$

we infer that $\operatorname{vol}_n(B_2^n) - \operatorname{vol}_n(P_N)$ is asymptotically not greater than

$$2^{1.802-2} n N^{-2/(n-1)} \operatorname{vol}_n(B_2^n).$$

(Note that, by our argument, we get that the above estimate holds for a subsequence of positive integers only.) Hence

$$\operatorname{del}_{n-1} \leqslant 2^{1.802-1} n \frac{\operatorname{vol}_n(B_2^n)}{(\operatorname{vol}_{n-1}(\partial B_2^n))^{(n+1)/(n-1)}} = \frac{2^{0.802}}{(\operatorname{vol}_{n-1}(\partial B_2^n))^{2/(n-1)}}. \quad \blacksquare$$

From Lemma 1, or more precisely, its proof we get the following result due to Gordon, Reisner, and Schütt.

THEOREM 4 [GRS]. *There are two positive constants c_7 and c_8 such that for every $n \geqslant 2$, and every $N \geqslant (c_8 n)^{(n-1)/2}$, and every polytope P_N contained in the Euclidean unit ball B_2^n with at most N vertices one has*

$$\operatorname{vol}_n(B_2^n) - \operatorname{vol}_n(P_N) \geqslant c_7 n \operatorname{vol}_n(B_2^n) N^{-2/(n-1)}.$$

Proof. From the proof of Lemma 1 we have

$$\operatorname{vol}_n(P_N) \leqslant \operatorname{vol}_n(B_2^n) \int_0^1 \left(1 - \frac{1}{2}\left(t \frac{n}{N} \cos\theta(1) \frac{\operatorname{vol}_n(B_2^n)}{\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{2/(n-1)}\right)^n dt.$$

Similarly as in the proof of Theorem 3, we argue that $\theta(1) \leqslant \frac{\pi}{4}$ in order to obtain

$$\operatorname{vol}_n(P_N) \leqslant \operatorname{vol}_n(B_2^n) \int_0^1 \left(1 - \frac{1}{2}\left(t \frac{n}{N} \frac{1}{\sqrt{2}} \frac{\operatorname{vol}_n(B_2^n)}{\operatorname{vol}_{n-1}(B_2^{n-1})}\right)^{2/(n-1)}\right)^n dt.$$

Hence, for

$$N \geqslant n^{(n-1)/2} \frac{n}{\sqrt{2}} \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}$$

we have

$$\left(t \frac{n}{N} \frac{1}{\sqrt{2}} \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)} \leqslant \frac{1}{n}.$$

Thus

$$\begin{aligned} \mathrm{vol}_n(P_N) &\leqslant \mathrm{vol}_n(B_2^n) \int_0^1 1 - c_9 \frac{n}{2} \left(t \frac{n}{N} \frac{1}{\sqrt{2}} \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)} dt \\ &= \mathrm{vol}_n(B_2^n) \left\{ 1 - c_9 \frac{n(n-1)}{2(n+1)} \left(\frac{n}{N} \frac{1}{\sqrt{2}} \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)} \right\} \\ &\leqslant \mathrm{vol}_n(B_2^n) \{ 1 - c_{10} n N^{-2/(n-1)} \}, \end{aligned}$$

for some numerical constants $c_9, c_{10} > 0$. ∎

Lemma 5. *Let $x_i \in \partial B_2^n$, $i = 1, ..., N$, and let Q_N be the intersection of all halfspaces $H^+(x_i)$ such that $B_2^n \subset H^+(x_i)$ and $x_i \in H(x_i)$.*

$$Q_N = \bigcap_{i=1}^{N} H^+(x_i).$$

Then we have

$$\mathrm{vol}_n(Q_N) - \mathrm{vol}_n(B_2^n) \geqslant \mathrm{vol}_n(B_2^n) \frac{n(n-1)}{2(n+1)} \left(\frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)}.$$

Proof. We may assume that Q_N is bounded, otherwise the inequality is trivial. In the proof of Lemma 1 we have established

$$\sin^{n-1} \theta(t) \geqslant t \frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})}.$$

Let σ be the normalized surface measure on ∂B_2^n and for $x \in \partial B_2^n$ let $R(x)$ be the distance from 0 to the point which is the intersection of ∂Q_N and the ray originating at 0 and passing through x. Then we have

$$\mathrm{vol}_n(Q_N) = \mathrm{vol}_n(B_2^n) \int_{\partial B_2^n} R^n(x) \, d\sigma(x).$$

We have that

$$R(x) = \frac{1}{\max_{1 \leqslant i \leqslant N} \langle x_i, x \rangle}.$$

Thus we get

$$\mathrm{vol}_n(Q_N) = \mathrm{vol}_n(B_2^n) \int_{\partial B_2^n} \left(\frac{1}{\max_{1 \leqslant i \leqslant N} \langle x_i, x \rangle} \right)^n d\sigma(x).$$

We have

$$\mathrm{vol}_{n-1}\{x \mid \max_{1 \leqslant i \leqslant N} \langle x, x_i \rangle \geqslant \cos \theta\} = t(\theta)\, \mathrm{vol}_{n-1}(\partial B_2^n).$$

Thus we get

$$\mathrm{vol}_n(Q_N) = \mathrm{vol}_n(B_2^n) \int_0^1 \cos^{-n}(\theta(t))\, dt = \int_0^1 (1 + \tan^2 \theta(t))^{n/2}\, dt.$$

Since

$$\sin \theta(t) \geqslant \left(t \frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{1/(n-1)}$$

we get

$$\begin{aligned}
\mathrm{vol}_n(Q_N) &\geqslant \mathrm{vol}_n(B_2^n) \int_0^1 \left(1 + \left(t \frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)} \right)^{n/2} dt \\
&\geqslant \mathrm{vol}_n(B_2^n) \int_0^1 1 + \frac{n}{2} \left(t \frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)} dt \\
&= \mathrm{vol}_n(B_2^n) + \mathrm{vol}_n(B_2^n) \frac{n(n-1)}{2(n+1)} \\
&\quad \times \left(\frac{n}{N} \cos \theta(1) \frac{\mathrm{vol}_n(B_2^n)}{\mathrm{vol}_{n-1}(B_2^{n-1})} \right)^{2/(n-1)}. \quad \blacksquare
\end{aligned}$$

THEOREM 6. *There are two positive constants c_{11} and c_{12} such that for every $n \geqslant 2$, and every $N \geqslant (c_{12} n)^{(n-1)/2}$, and every polytope Q_N which has at most N facets and is contained in the Euclidean ball B_2^n*

$$\mathrm{vol}_n(B_2^n) - \mathrm{vol}_n(Q_N) \geqslant c_{11} n\, \mathrm{vol}_n(B_2^n)\, N^{-2/(n-1)}.$$

The proof of Theorem 6 is parallel to that of Theorem 4 and is left to the reader.

The order of magnitude of the constant $c_{11}n$ is optimal, i.e., the constant is linear in n. Indeed, the following proposition is a consequence of a result in [BI] and can be found in [RSW].

PROPOSITION 7. *There exists a constant c_0 such that for all n, for every convex body C in $\mathbb{R}^n$ which is contained in B_2^n and for $N > c_{13}^{(n-1)/2}$, there exists a convex polytope $P \subset C$ with no more than N vertices, such that*

$$d_H(P, C) \leqslant \frac{c_{13}}{N^{2/(n-1)}}.$$

For $C = B_2^n$ we get

$$\left(1 - \frac{c_{13}}{N^{2/(n-1)}}\right) B_2^n \subset P \subset B_2^n$$

and by dualizing

$$B_2^n \subset P^* \subset \left(1 - \frac{c_{13}}{N^{2/(n-1)}}\right)^{-1} B_2^n.$$

P^* has N facets. Hence

$$\mathrm{vol}_n(B_2^n) \leqslant \mathrm{vol}_n(P^*) \leqslant \mathrm{vol}_n(B_2^n) \left(1 - \frac{c_{13}}{N^{2/(n-1)}}\right)^{-n}.$$

Therefore, for sufficiently large N

$$\mathrm{vol}_n(B_2^n) \leqslant \mathrm{vol}_n(P^*) \leqslant \mathrm{vol}_n(B_2^n) \left(1 + \frac{c_{14} n}{N^{2/(n-1)}}\right).$$

REFERENCES

[BI] E. M. Bronshteyn and L. D. Ivanov, The approximation of convex sets by polyhedra, *Siberian Math. J.* **16** (1975), 1110–1112.

[Ed] H. Edelsbrunner, "Algorithms in Combinatorial Geometry," Springer-Verlag, Berlin, 1987.

[GRS] Y. Gordon, S. Reisner, and C. Schütt, Umbrellas and polytopal approximation of the euclidean ball, *J. Approx. Theory* **90** (1997), 9–22.

[GR] P. M. Gruber, Asymptotic estimates for best and stepwise approximation of convex bodies II, *Forum Math.* **5** (1993), 521–538.

[KL] G. A. Kabatjanskii and V. I. Levenstein, Bounds for packings on a sphere and in space, *Problems Inform. Transmission* **14** (1978), 1–17.

[Mac] A. M. Macbeath, An extremal property of the hypersphere, *Proc. Cambridge Philos. Soc.* **47** (1951), 245–247.

[RSW] S. Reisner, C. Schütt, and E. Werner, Dropping a vertex or a facet from a convex polytope, *Fund. Math.*, to appear.